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THE STATIC STATE OF A TWO-PHASE SOLID MIXTURE IN A STRESSED ELASTIC BAR

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Abstract—We consider the equilibrium state and phase separation of a stressed elastic cylindrical bar which is composed of a mixture of two materials of fixed amounts. The Young's modulus of the mixture is a function of the local volume fraction, which itself depends on the position in the bar. We determine the distribution of the two pure components and the field of internal strain throughout the bar so that the total potential energy is minimized. The problem is one of nonconvex minimization, and because of this we find that the two materials which comprise the bar are separated in the stressed equilibrium state. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

When a body which is composed of many components is stressed, there is potential for diffusion of one component through the other and separation may take place. For example, the moisture content of concrete, bricks, and other masonry materials is affected by the state of stress in the material and this fact is an important element in the list of criteria for design. Degradation and failure are issues of great concern not only to the structural designer but also to the machine designer and to the metallurgist who designs alloys. The separation of materials can aggravate these matters, and so, in many cases, stabilizers, non-wetting agents, etc., are included in the mixture. On the other hand, in many problems of optimal design the idea is to put together a certain set of objects in such a way that a given design feature is optimized. This may require the objects to be homogenized into one mixture or it may require that they preserve their distinctness, but be "well-placed". There are examples at both ends of the spectrum, and in many cases it is not an easy problem to solve.

In the present work, we consider a binary mixture of two elastic materials, each having known properties in the pure state, in a body of fixed total concentration. The fundamental question which we address is how the materials should be arranged so that the total potential energy is minimized. If one thinks of this as a bi-component mixture, perhaps a metal alloy or a concrete with a variable moisture content, which is subjected to an equilibrium loading system, then we wish to determine the internal fields of strain and concentration so that the total potential energy is minimized. In this case, we are interested to know whether or not the component materials will be segregated in the optimal minimum energy state. We are not considering the process of diffusion in this work and the related interesting question of how the material components move in time in relation to the changing internal deformation field under a fixed loading system. In fact, it is not presently known whether the diffusion process in this case would evolve towards some other weakly stable state of equilibrium.

The minimization problem of this work is nonconvex even though at any fixed concentration the strain-energy function is quadratic so that the stress-strain behavior is linear. The elasticity of the material (as well as the mass density) is assumed to depend on the volumetric concentration c, which is confined to the interval [0,1]. As the concentration changes, the material becomes more or less stiff and as a result of this the strain-energy-concentration graph is not convex.

In Section 2, we introduce the problem of an elastic bar composed of a binary mixture which is given an axial displacement and subject to an axial body force field. When the body force is non-zero, there is a bias on the loading system which gives rise to a helpful uniqueness property. The nonconvex nature of the problem is identified and preliminary matters are discussed.

In Section 3, we give a brief review of the analysis in the elementary case when the body force field vanishes. There is a high degree of nonuniqueness in the structure of a minimizer in this case, but all of the structures can be characterized easily, and we find that they associate with boundary points at $c = 0$ or $c = 1$ of the strain-energy-concentration function. Then in Section 4 we consider the main case when the body force does not vanish. Here, the problem is somewhat harder to solve. The presence of a body force field induces a regularity and uniqueness on the solution which predicts that the component materials become segregated into at least two and at most three intervals of concentration c equal to either 0 or 1. In these intervals the stress and strain fields are smooth and not constant. A main feature of the solution is that the stiffer of the two materials (i.e., the material with the larger modulus of elasticity in its pure state) becomes located in that interval of the bar where the absolute value of the stress is the smallest.

In Section 5, we give a graphical representation of the minimizing state in the bar as a function of the axial end displacement for a fixed body force and fixed total concentration. We show that in the state of minimum potential energy the materials are separated into intervals and we illustrate how the stress varies in each interval.

2. PRELIMINARIES: STATEMENT OF THE PROBLEM

Let $B = (0, L)$ denote the undistorted natural reference configuration of a bar of length L and uniform cross sectional area A. The particles of the bar are distinguished only by the cross section in which they lie and we shall denote each cross section by $x \in B \subset \mathbb{R}$. The bar is supposed to consist of an axially distributed mixture of two elastic materials, components "0" and "1", and we let $c(x) \in [0,1]$ denote the *specific volume fraction* of component "1" per unit volume of B at the cross section x. Thus,

$$V_1 = A\int_0^L c(x)\,\mathrm{d}x, \quad V_0 = AL - V_1, \tag{2.1}$$

represent the total volumes of the two materials in B. We shall let ρ_0 and ρ_1 denote the *mass density* of the two pure components which are contained in B, so that the mass density of a mixture whose specific volume fraction is c is given by:

$$\rho = \hat{\rho}(c) = \rho_1 c + \rho_0(1-c). \tag{2.2}$$

Therefore, at any cross section $x \in B$, the mixture mass density is given by $\hat{\rho}(c(x))$.

By a deformation of B we mean a continuous 1–1 mapping $y(\cdot) : B \to \mathbb{R}$ with $y'(\cdot) > 0$. The associated *displacement* field $u(\cdot) : B \to \mathbb{R}$ is then given by

$$u(x) = y(x) - x \tag{2.3}$$

for all $x \in B$, and the corresponding *strain* field is $e(\cdot) = u'(\cdot) > -1$.

The bar B is considered to be an elastic solid for which the specific strain energy per unit reference volume has the classical quadratic form

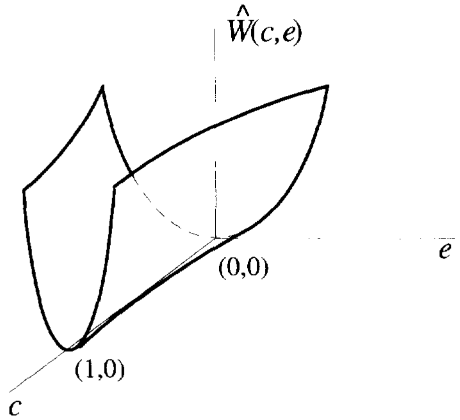

Fig. 1. Form of the specific stored energy function.

$$W = \tfrac{1}{2}Ee^2, \quad E > 0.$$

Here, however, we assume that the *modulus of elasticity* for the mixture, $E = \hat{E}(c)$, is a smooth, positive, monotone increasing function of the volume fraction, and that

$$\hat{E}'(c) > 0, \quad (\ln(\ln \hat{E}(c)))'' < 0 \tag{2.4}$$

for all $c \in [0, 1]$. We shall let $E_0 = \hat{E}(0)$ and $E_1 = \hat{E}(1) > E_0$ be the moduli of elasticity for the respective pure components; component "1" is thus the "stiffer" of the two materials. In this case, if (2.4) holds, the strain energy function

$$W = \hat{W}(c, e) = \tfrac{1}{2}\hat{E}(c)e^2 \tag{2.5}$$

is nonconvex and of the form illustrated in Fig. 1. For convenience we assume that $\hat{W}(c, e)$ is defined in (2.5) for all $e \in \mathbb{R}$ even though physical considerations require that the strain e be bounded below by -1. Thus, the domain of definition of $\hat{W}(\cdot,\cdot)$ is $[0, 1] \times \mathbb{R}$.

Concerning the imposed actions, the end of B at $x = 0$ is fixed, while the end $x = L$ is given a prescribed displacement Δ. In addition, gravity is supposed to act on the bar in the increasing axial direction. We are interested in determining the deformation of the bar and the distribution of the two components in B, i.e., the specific volume fraction $c(x)$ for $x \in B$, so that the total potential energy is minimized. The total volume V_1 of component "1" is given, and thus by $(2.1)_2$ so also is V_0, the total volume of component "0". If we let g denote the standard gravitational constant, since the cross sectional area of B is uniform, then the total potential energy is given by

$$\mathsf{E}(c, u) = \int_0^L [\hat{W}(c(x), u'(x)) - g\hat{\rho}(c(x))u(x)]\,\mathrm{d}x, \tag{2.6}$$

where $\hat{W}(\cdot,\cdot)$ has the form (2.5) and $\hat{\rho}(\cdot)$ is defined in (2.2). The problem then is to

$$\text{minimize}\, \mathsf{E}(c, u)(c(\cdot), u(\cdot)) \in \mathscr{A}, \tag{2.7}$$

where the class of admissible functions $\mathscr{A}$ is given by

$$\mathscr{A} = \Big\{c(\cdot)\in L^{\infty}(B),\quad u(\cdot)\in W^{1,2}(B)|\quad c(x)\in[0,1]\forall x\in B,$$

$$\int_0^L c(x)\,\mathrm{d}x = Q \equiv V_1/A \leqslant L,\quad u(0)=0,\quad u(L)=\Delta\Big\}. \tag{2.8}$$

This problem is a nonconvex minimization problem because of the form of $\hat{W}(\cdot,\cdot)$. Moreover, we emphasize that $\hat{W}(c,e)$ is defined only for c in the closed interval [0, 1], and so the edges of $\hat{W}(\cdot,\cdot)$ at $c = 0$ and $c = 1$, as illustrated in Fig. 1, are expected to play an important part in the following analysis.

3. THE ELEMENTARY CASE: ABSENCE OF BODY FORCE

In this section, we consider for completeness the case in which the body force is negligible, which mathematically is equivalent to setting $g = 0$ in (2.6). Then (2.6) reduces to

$$E(c,u) = \int_0^L \hat{W}(c(x),u'(x))\,\mathrm{d}x. \tag{3.1}$$

The results here are not new and are contained in the work of Dunn and Fosdick (1980), for example. It is well known from the Weierstrass condition,† that if $(\tilde{c}(\cdot), \tilde{u}(\cdot))\in\mathscr{A}$ is a minimizer of $E(c,u)$, then at any point $x\in B$ of smoothness of $\tilde{c}(\cdot)$ and $\tilde{u}(\cdot)$,

$$\hat{W}(\tilde{c}(x),\tilde{u}'(x)) \leqslant \alpha\hat{W}(c_1,e_1)+(1-\alpha)\hat{W}(c_2,e_2),$$

for all $\alpha\in(0,1)$ and all (c_1, e_1) and (c_2, e_2) in the domain of definition of $\hat{W}(\cdot,\cdot)$ such that

$$\alpha(c_1,e_1)+(1-\alpha)(c_2,e_2) = (\tilde{c}(x),\tilde{u}'(x)).$$

In words, the range of a minimizer may include only those points in the domain of definition of $\hat{W}(\cdot,\cdot)$ that correspond to lower support points of the surface $W = \hat{W}(c,e)$. Because of the nonconvex form of $\hat{W}(\cdot,\cdot)$, as illustrated in Fig. 1, this implies that $(\tilde{c}(\cdot),\tilde{u}(\cdot))$ can take values only on the two boundaries of the domain of definition of $\hat{W}(\cdot,\cdot)$, i.e., $\{(c,e)|\ c=0$ and $e\in\mathbb{R}$ or $c=1$ and $e\in\mathbb{R}\}$.

Now suppose that $W^{\#}(\cdot,\cdot)$ is the lower convex envelope of $\hat{W}(\cdot,\cdot)$. Then, for $\Delta\neq0$, the tangent plane to $W^{\#}(\cdot,\cdot)$ at the point

$$P\equiv\left(\frac{Q}{L},\ \frac{\Delta}{L},\ W^{\#}\left(\frac{Q}{L},\ \frac{\Delta}{L}\right)\right)$$

will support the surface $\hat{W}(\cdot,\cdot)$ from below at two unique points $P_1\equiv(1, e_1, \hat{W}(1,e_1))$ and $P_0\equiv(0, e_0, \hat{W}(0,e_0))$ as shown in Fig. 2.

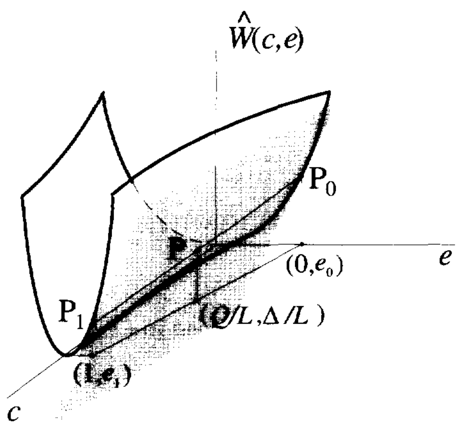

Fig. 2. Tangent plane to the lower convex envelope of $\hat{W}(\cdot,\cdot)$ and the support points P_0 and P_1.

† See, for example, Dunn and Fosdick (1980), Theorem 1.

Clearly, the three points P_1, P, P_0 all lie on the same straight line and so we may write

$$P = \lambda P_1 + (1-\lambda) P_0,$$

for some $\lambda \in [0, 1]$. In fact, this implies that

$$\lambda = \frac{Q}{L} = \frac{\Delta/L - e_0}{e_1 - e_0}, \tag{3.2}$$

and

$$W^{\#}\left(\frac{Q}{L}, \frac{\Delta}{L}\right) = \lambda \hat{W}(1, e_1) + (1-\lambda)\hat{W}(0, e_0). \tag{3.3}$$

We shall now show that a minimizer $(\tilde{c}(\cdot), \tilde{u}(\cdot))$ of $E(c, u)$ in $\mathscr{A}$ may be constructed of the form

$$(\tilde{c}(x), \tilde{u}'(x)) = \begin{cases} (0, e_0) & \forall x \in B_0 = B \backslash B_1, \\ (1, e_1) & \forall x \in B_1, \end{cases} \tag{3.4}$$

where the Lebesgue measure of the set $B_1 \subset B$ is given by

$$\mu(B_1) = \lambda L. \tag{3.5}$$

The claim of (3.5) is straightforward because $\tilde{u}(\cdot) \in \mathscr{A}$ must satisfy $\tilde{u}(0) = 0$ and $\tilde{u}(L) = \Delta$. Therefore,

$$\int_0^L \tilde{u}'(x)\,\mathrm{d}x = \Delta = \int_{B\backslash B_1} e_0\,\mathrm{d}x + \int_{B_1} e_1\,\mathrm{d}x = e_0(L - \mu(B_1)) + e_1 \mu(B_1),$$

and this, along with (3.2), implies (3.5).

To see that (3.4) is the basic structure of a minimizer, we observe, using (3.3) and (3.5), that

$$\begin{aligned} \int_0^L \hat{W}(\tilde{c}(x), \tilde{u}'(x))\,\mathrm{d}x &= \int_{B\backslash B_1} \hat{W}(0, e_0)\,\mathrm{d}x + \int_{B_1} \hat{W}(1, e_1)\,\mathrm{d}x \\ &= \hat{W}(0, e_0)(L - \mu(B_1)) + \hat{W}(1, e_1)\mu(B_1) \\ &= L[\lambda \hat{W}(1, e_1) + (1-\lambda)\hat{W}(0, e_0)] \\ &= L W^{\#}\left(\frac{Q}{L}, \frac{\Delta}{L}\right). \end{aligned} \tag{3.6}$$

But the lower convex envelope $W^{\#}(\cdot,\cdot)$ is convex and has the property that

$$W^{\#}(c, e) \geqslant W^{\#}\left(\frac{Q}{L}, \frac{\Delta}{L}\right) + \left(c - \frac{Q}{L}\right) W^{\#}_{,c}\left(\frac{Q}{L}, \frac{\Delta}{L}\right) + \left(e - \frac{\Delta}{L}\right) W^{\#}_{,e}\left(\frac{Q}{L}, \frac{\Delta}{L}\right), \tag{3.7}$$

for all $(c, e) \in [0, 1] \times \mathbb{R}$, where $W^{\#}_{,c}(\cdot,\cdot)$ and $W^{\#}_{,e}(\cdot,\cdot)$ denote the respective partial derivatives of $W^{\#}(\cdot,\cdot)$. Thus if we introduce an arbitrary pair of functions $(c, u) \in \mathscr{A}$, replace (c, e) in (3.7) by $(c(x), u'(x))$ and integrate, it then follows that

$$\int_0^L W^{\#}(c(x),u'(x))\,\mathrm{d}x \geqslant LW^{\#}\left(\frac{Q}{L},\ \frac{\Delta}{L}\right), \tag{3.8}$$

because

$$\int_0^L \left(c(x)-\frac{Q}{L}\right)\mathrm{d}x = 0, \quad \int_0^L \left(u'(x)-\frac{\Delta}{L}\right)\mathrm{d}x = 0,$$

for all $(c(\cdot),u(\cdot))\in\mathscr{A}$.

Finally, because $W^{\#}(\cdot,\cdot)$ is the lower convex envelope of $\hat{W}(\cdot,\cdot)$, we note from (3.6) and (3.8) that

$$\int_0^L W^{\#}(\tilde{c}(x),\tilde{u}'(x))\,\mathrm{d}x \leqslant \int_0^L W^{\#}(c(x),u'(x))\,\mathrm{d}x \leqslant \int_0^L \hat{W}(c(x),u'(x))\,\mathrm{d}x$$

for all $(c(\cdot),u(\cdot))\in\mathscr{A}$. Since it is straightforward to construct a pair of functions $(c(\cdot), u(\cdot))\in\mathscr{A}$ from the basic form of (3.4), (3.5) and (3.2), we conclude that $(\tilde{c}(\cdot),\tilde{u}(\cdot))\in\mathscr{A}$ is a minimizer for the elementary case in which the body force is negligible. Of course, a minimizer is not unique because rearrangements of the field constructed above are allowed, as long as the measure $\mu(B_1)$ in (3.5) is fixed at the value λL.

4. THE MAIN CASE: PRESENCE OF BODY FORCE

We now turn to the general minimization problem of this work, defined in (2.6), (2.7) and (2.8). Here, it is helpful to first consider the associated relaxed problem which is defined by replacing the stored energy function $\hat{W}(\cdot,\cdot)$ appearing in (2.6) by its lower convex envelope $W^{\#}(\cdot,\cdot)$. With the aid of (2.4) and (2.5), this lower convex envelope is easily constructed and has the form

$$W^{\#}(c,e) = \tfrac{1}{2}E^{\#}(c)e^2, \tag{4.1}$$

where

$$E^{\#}(c) = \frac{E_0E_1}{E_1-c(E_1-E_0)}. \tag{4.2}$$

We notice that $E^{\#}(c)$ is the weighted harmonic mean of the elastic moduli $E_0 = \hat{E}(0)$ and $E_1 = \hat{E}(1) > E_0$. Then, the *relaxed problem* may be stated as follows:

$$\underset{(c(\cdot),u(\cdot))\in\mathscr{A}}{\text{minimize}}\ \boldsymbol{E}^{\#}(c,u), \tag{4.3}$$

where

$$\boldsymbol{E}^{\#}(c,u) = \int_0^L [W^{\#}(c(x),u'(x))-g\hat{\rho}(c(x))u(x)]\,\mathrm{d}x. \tag{4.4}$$

The class $\mathscr{A}$ is defined in (2.8), and the form of $\hat{\rho}(\cdot)$ is given in (2.2).

We shall now construct the unique† minimizer $(\tilde{c}(\cdot),\tilde{u}(\cdot))\in\mathscr{A}$ for this relaxed problem and show that it has the additional important property that

† Here, we mean unique in the sense of almost everywhere (a.e.) in B. That is, modulo adjustments on sets of Lebesgue measure zero.

$$E^{\#}(\tilde{c}, \tilde{u}) = E(\tilde{c}, \tilde{u}). \tag{4.5}$$

That is, we shall show that the field $(\tilde{c}(\cdot), \tilde{u}(\cdot)) \in \mathscr{A}$ which solves the relaxed minimization problem not only takes on values that are in the domain of definition of the original (unrelaxed) problem, but also that the total potential energy of the original problem based upon this field is the *same* as the (minimal) total potential energy of the relaxed problem. Then, because for any $(c(\cdot), u(\cdot)) \in \mathscr{A}$ we know that

$$E^{\#}(\tilde{c}, \tilde{u}) \leqslant E^{\#}(c, u) \leqslant E(c, u),$$

the latter inequality arising from the fact that $W^{\#}(\cdot,\cdot)$ is the lower convex envelope of $\hat{W}(\cdot,\cdot)$, we may conclude that $(\tilde{c}(\cdot), \tilde{u}(\cdot))$ is a minimizer of the original problem (2.6), (2.7), (2.8). In fact, $(\tilde{c}(\cdot), \tilde{u}(\cdot))$ is the *unique* minimizer of the original problem because if such is not the case, then there must be another pair $(\hat{c}(\cdot), \hat{u}(\cdot)) \in \mathscr{A}$ for which $E(\hat{c}, \hat{u}) = E(\tilde{c}, \tilde{u})$. But then, $\hat{c}(x)$ is either 0 or 1 almost everywhere in B because, as a consequence of the Weierstrass condition, only the lower support points for $\hat{W}(\cdot,\cdot)$ can be in the range of a minimizer almost everywhere in B. In this case, $(\hat{c}(\cdot), \hat{u}(\cdot))$ is in the domain of definition of the relaxed problem and, therefore, we see that $E(\hat{c}, \hat{u}) = E^{\#}(\hat{c}, \hat{u})$, which, in turn, using (4.5), gives $E^{\#}(\hat{c}, \hat{u}) = E^{\#}(\tilde{c}, \tilde{u})$. Because $(\tilde{c}(\cdot), \tilde{u}(\cdot))$ is the unique minimizer of the relaxed problem, we conclude that $(\tilde{c}(\cdot), \tilde{u}(\cdot)) = (\hat{c}(\cdot), \hat{u}(\cdot))$ modulo a trivial adjustment in B on sets of Lebesgue measure zero.

First let us consider briefly some general features of the relaxed problem (4.1)–(4.4) which lead to the conclusion that a minimizer does indeed exist. To do this, it is convenient to introduce

$$\varphi(x) = \int_0^x c(s)\,\mathrm{d}s, \tag{4.6}$$

and to use (2.2) to integrate the second term in (4.4) by parts. Because $(c(\cdot), u(\cdot)) \in \mathscr{A}$, we find that up to an additive constant the functional $E^{\#}(\cdot,\cdot)$ may be replaced by

$$\bar{E}^{\#}(\varphi, u) \equiv \int_0^L [W^{\#}(\varphi'(x), u'(x)) + gu'(x)(x\rho_0 + (\rho_1 - \rho_0)\varphi(x))]\,\mathrm{d}x. \tag{4.7}$$

Thus, the relaxed minimization problem can be carried out, *equivalently*, on the functional (4.7) over the class of functions $(\varphi(\cdot), u(\cdot)) \in (\varphi_0(\cdot), u_0(\cdot)) + \bar{\mathscr{A}}_0$, where

$$\bar{\mathscr{A}}_0 = \{\bar{\varphi}(\cdot) \in W_0^{1,\infty}(B), \bar{u}(\cdot) \in W_0^{1,2}(B) |\quad \bar{\varphi}'(x) \in [-Q/L, 1 - Q/L]\ \forall\, x \in B\}, \tag{4.8$_1$}$$

and

$$\varphi_0(x) = \frac{Q}{L}x, \quad u_0(x) = \frac{\Delta}{L}x. \tag{4.8$_2$}$$

For every $x \in B$ with $Q \leqslant L$, we define the *norm* of $(\varphi(\cdot), u(\cdot))$ in either $\bar{\mathscr{A}}_0$ or $(\varphi_0(\cdot), u_0(\cdot)) + \bar{\mathscr{A}}_0$ as $\|(\varphi, u)\| = \|\varphi\|_{W^{1,\infty}(B)} + \|u\|_{W^{1,2}(B)}$ and we suppose the *topology* τ to be associated with *weak** convergence for functions $\psi(\cdot) \in W^{1,\infty}(B)$ and the *weak* convergence for functions $v \in W^{1,2}(B)$. Then, according to a fundamental theorem in the calculus of variations (see, e.g., Buttazzo (1989)) the infimum of $\bar{E}^{\#}(\varphi, u)$ over all $(\varphi(\cdot), u(\cdot)) \in (\varphi_0(\cdot), u_0(\cdot)) + \bar{\mathscr{A}}_0$ attains its minimum† with respect to the topology τ. For the sake of continuity, we shall not digress now, but rather give a brief discussion of this matter in Appendix A.

† We also have shown this using a classical direct approach in the note of Fosdick *et al.* (1995).

We now return to the construction of the (unique) minimizer of the relaxed problem: i.e., determine $(\tilde{c}(\cdot), \tilde{u}(\cdot)) \in \mathscr{A}$ such that

$$E^{\#}(\tilde{c}, \tilde{u}) = \underset{(c(\cdot), u(\cdot)) \in \mathscr{A}}{\text{minimum}} E^{\#}(c, u), \tag{4.9}$$

where $E^{\#}(c, u)$ is defined in (4.4) and $\mathscr{A}$ is defined in (2.8). As a first step, we wish to show that $\tilde{c}(\cdot)$ can only take on the values 0 or 1 almost everywhere in the body B. In fact, suppose not and let $x_0 \in B$ be such that $\tilde{c}(x) \in]0,1[$ almost everywhere in a neighborhood $\mathscr{N}(x_0)$ of x_0. Then, from (4.1), (4.2), (4.4), (2.2) and (2.8) we readily find that the relevant system of Euler equations may be written as

$$E^{\#}(\tilde{c}(x))\tilde{u}'(x) + g\int_0^x [\rho_1 \tilde{c}(s) + \rho_0(1-\tilde{c}(s))]\,ds = K \quad \text{a.e.}\ x \in B, \tag{4.10$_1$}$$

and

$$\frac{E_1 - E_0}{2E_1 E_0}[E^{\#}(\tilde{c}(x))\tilde{u}']^2 - g(\rho_1 - \rho_0)\tilde{u}(x) = J \quad \text{a.e.}\ x \in \mathscr{N}(x_0), \tag{4.10$_2$}$$

where K and J are constants. It is clear that $(\tilde{c}(\cdot), \tilde{u}(\cdot)) \in \mathscr{A}$ may be adjusted on a set of Lebesgue measure zero so that $(4.10)_1$ applies everywhere in B without changing (4.9). Thus, we see that $E^{\#}(\tilde{c}(x))\tilde{u}'(x)$ is continuous for all $x \in B$, and the second combined with the first of (4.10) implies that the weak derivative of $\tilde{u}(\cdot)$ satisfies

$$g(\rho_1 - \rho_0)\tilde{u}'(x) = -g\frac{E_1 - E_0}{E_1 E_0}E^{\#}(\tilde{c}(x))\tilde{u}'(x)[\rho_1 \tilde{c}(x) + \rho_0(1-\tilde{c}(x))] \quad \text{a.e.}\ x \in \mathscr{N}(x_0).$$

Then, using (4.2) we find, after an elementary simplification, that

$$(E_1\rho_1 - E_0\rho_0)\tilde{u}'(x) = 0 \quad \text{a.e.}\ x \in \mathscr{N}(x_0).$$

Thus, if $E_1\rho_1 \neq E_0\rho_0$ we see that $\tilde{u}'(x) = 0$ for almost all $x \in \mathscr{N}(x_0)$ and this together with $(4.10)_1$ implies that

$$\int_0^x \tilde{c}(s)\,ds = \text{const.} \quad \text{a.e.}\ x \in \mathscr{N}(x_0).$$

Consequently, we conclude that $\tilde{c}(x) = 0$ for almost all $x \in \mathscr{N}(x_0)$, but this contradicts the previous hypothesis that $\tilde{c}(x) \in]0,1[$ almost everywhere in $\mathscr{N}(x_0)$.

For definiteness, in the remainder of this work we shall *assume* that

$$\frac{E_1\rho_1}{E_0\rho_0} > 1, \tag{4.11}$$

and then, because of the argument given above, we know that $\tilde{c}(x) = 0$ or 1 almost everywhere in B. In fact, there must be at least some interval of B in which $\tilde{c}(x) = 1$ almost everywhere; otherwise $\tilde{c}(x) = 0$ almost everywhere in B and the integral constraint on $\tilde{c}(\cdot)$ in $\mathscr{A}$ cannot be satisfied. Let B_1 denote the set of all maximal open sub-intervals of B for which $\tilde{c}(x) = 0$ almost everywhere. Except for points of Lebesgue measure zero in B we have $B = B_0 \cup B_1$. In B_1, of course, there may be sets of measure zero on which $\tilde{c}(x) \in [0,1)$, and in B_0 there may be sets of measure zero on which $\tilde{c}(x) \in (0,1]$. However, by adjusting $\tilde{c}(\cdot)$ on these meaningless sets of Lebesgue measure zero we can assume, without loss of generality, that $B = B_0 \cup B_1$, where B_0 and B_1 are composed of *sets of intervals* such that

$$\tilde{c}(x) = \begin{cases} 0 & \forall x \in B_0 \\ 1 & \forall x \in B_1. \end{cases} \tag{4.12}$$

Thus, a minimizer of the relaxed problem (4.9) will satisfy the property (4.5) and so it will also be a minimizer of the original problem (2.6), (2.7), (2.8).

The Euler equation $(4.10)_2$ no longer applies since it followed from the assumption that $\tilde{c}(\cdot)$ could take on values strictly between 0 and 1 almost everywhere in some measurable sub-part of B. Now, however, due to the argument leading to (4.12), only "one-sided" variations of $\tilde{c}(\cdot)$ in $\mathscr{A}$ are admissible. Thus, using (4.1), (4.2), (4.4), (2.2), (2.8), and a standard variational argument, we find that the left hand side of $(4.10)_2$ cannot be less anywhere in B_0 than it is anywhere in B_1, i.e.,

$$\left[\frac{E_1-E_0}{2E_1E_0}[E_0\tilde{u}']^2 - g(\rho_1-\rho_0)\tilde{u}(x)\right]_{x\in B_0} \geqslant \left[\frac{E_1-E_0}{2E_1E_0}[E_1\tilde{u}']^2 - g(\rho_1-\rho_0)\tilde{u}(x)\right]_{x\in B_1}. \tag{4.13}$$

Our aim is now to determine the unique solution of $(4.10)_1$ and (4.13), in the class $\mathscr{A}$ of (2.8), which respects the condition (4.12) and the inequality (4.11). To do this, it is first helpful to observe that $(4.10)_1$, (4.12) and (4.2) imply

$$E_0\tilde{u}''(x) + g\rho_0 = 0 \quad \forall x \in B_0, \tag{4.14$_1$}$$

and

$$E_1\tilde{u}''(x) + g\rho_1 = 0 \quad \forall x \in B_1. \tag{4.14$_2$}$$

Thus, if $x^* \in B$ is any point of separation between B_0 and B_1, we find, by integrating $(4.14)_1$, that for all x in a neighborhood of x^* in B_0,

$$\tilde{u}(x) = \tilde{u}(x^*) + \tilde{u}_0'(x^*)(x-x^*) - \frac{g\rho_0}{2E_0}(x-x^*)^2,$$

where $\tilde{u}_0'(x^*)$ is the limiting value of $\tilde{u}'(x)$ as x tends to x^* from within B_0. With this, the left hand side of inequality (4.13) readily reduces to

$$\frac{E_1-E_0}{2E_1E_0}[E_0\tilde{u}_0'(x^*)]^2 - g(\rho_1-\rho_0)\tilde{u}(x^*) + g\frac{E_1\rho_1-E_0\rho_0}{E_0E_1}\left[\frac{g\rho_0}{2}(x-x^*)^2 - E_0\tilde{u}_0'(x^*)(x-x^*)\right].$$

Consequently, after evaluating the right hand side of (4.13) in the limit $x \to x^*$, and by using the above expression on the left hand side, together with the fact that $E^{\#}(\tilde{c}(x))\tilde{u}'(x)$ is continuous in B, we see from (4.13) that

$$g\frac{E_1\rho_1-E_0\rho_0}{E_0E_1}\left[\frac{g\rho_0}{2}(x-x^*)^2 - E_0\tilde{u}_0'(x^*)(x-x^*)\right] \geqslant 0$$

for all $x \in B_0$ in a neighborhood of x^*. Because of inequality (4.11), we then find that a minimizer of the relaxed problem (4.9) must satisfy the monotonicity conditions

$$E_0\tilde{u}_0'(x^*)\begin{cases} >0 \Rightarrow B_0 \ll B_1, \\ <0 \Rightarrow B_1 \ll B_0. \end{cases} \tag{4.15}$$

In words, *if $E^{\#}(\tilde{c}(x))\tilde{u}'(x)$ is positive (negative), then the local mutual ordering of B_0 and B_1 at x^* is $B_0 \ll B_1$ ($B_1 \ll B_0$) and $\tilde{c}(x)$ changes from* 0 *to* 1 (1 *to* 0) *as x increases through x^*.*

Similar to the above argument, we may use $(4.14)_2$ to find that

$$\tilde{u}(x) = \tilde{u}(x^*) + \tilde{u}_1'(x^*)(x-x^*) - \frac{g\rho_1}{2E_1}(x-x^*)^2$$

for all $x \in B_1$ in a neighborhood of x^*, where $\tilde{u}_1'(x^*)$ is the limiting value of $\tilde{u}'(x)$ as x tends to x^* from within B_1. Then, by using this expression in the right hand side of (4.13) and evaluating the left hand side in the limit $x \to x^*$, we find that

$$g\frac{E_1\rho_1 - E_0\rho_0}{E_0E_1}\left[\frac{g\rho_1}{2}(x-x^*)^2 - E_1\tilde{u}_1'(x^*)(x-x^*)\right] \leqslant 0$$

for all $x \in B_1$ in a neighborhood of x^*. Since we know that $E^{\#}(\tilde{c}(x))\tilde{u}'(x)$ is continuous in B, so that $E_0\tilde{u}_0'(x^*) = E_1\tilde{u}_1'(x^*)$, we again recover the monotonicity condition (4.15) from this inequality. However, in addition, we see that $E_0\tilde{u}_0'(x^*) = E_1\tilde{u}_1'(x^*) \neq 0$ *at any point* $x^* \in B$ *of separation between* B_0 *and* B_1.

In general, these last considerations show that *a minimizer of the relaxed problem* (4.9) *can have at most two points in B where* B_0 *is separated from* B_1. For example, suppose that $x \in B_0$, so that $\tilde{c}(x) = 0$ and that $E_0\tilde{u}'(x)$ is strictly positive. According to $(4.14)_1$, $E_0\tilde{u}'(x)$ must be a decreasing function of x and because of (4.15) the domain B_0 must then persist clear back to $x = 0$. As x increases, however, there may be a point x^* at which B_0 and B_1 are separated. As this point is passed, $\tilde{c}(x)$ must change from 0 to 1, according to (4.15), $E_0\tilde{u}'(\cdot)$ must be *positive*, and $E_0\tilde{u}'(\cdot)$ must change continuously to $E_1\tilde{u}'(\cdot)$. As x continues to increase, now in B_1, we see from $(4.14)_2$ that $E_1\tilde{u}'(x)$ must decrease. There may or may not be another point x^{**} at which B_0 and B_1 are again separated. If there is, then as this point is passed, $\tilde{c}(x)$ must change back from 1 to 0 and, according to $(4.15)_2$, $E_1\tilde{u}'(\cdot)$ must change continuously to $E_0\tilde{u}'(\cdot)$, which now must be *negative*. As x continues to increase (now in B_0), we see from $(4.14)_1$ that $E_0\tilde{u}'(x)$ must decrease, and because of (4.15) there can be no further points in B which separate B_0 and B_1. A trace of this type of minimizer for the relaxed problem (4.9) is indicated on the graph of the stored density energy function $\hat{W}(\cdot,\cdot)$ in Fig. 3. Notice that the minimizer associates with points only on the edges $c = 0$ and $c = 1$ of the stored energy function. Thus, the property (4.5) is indeed satisfied and consequently the minimizer will also serve to minimize the original problem (2.6), (2.7), (2.8).

In all cases, the set $B_1 \subset B$ where $\tilde{c}(x) = 1$ must be connected, and must be of length $Q \leqslant L$ (recall (2.8)). Also, if we define the "stress"

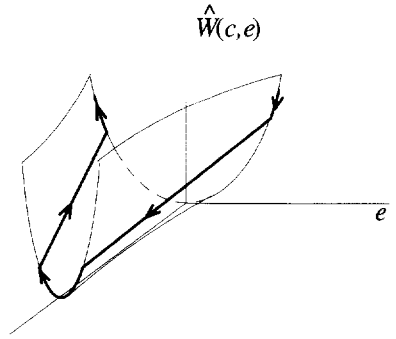

Fig. 3. Trace of a minimizer for which B_0 and B_1 are separated by two points. Arrows indicate increasing values of $x \in B$.

$$\tilde{\sigma}(x) \equiv E^{\#}(\tilde{c}(x))\tilde{u}'(x) = \begin{cases} E_0\tilde{u}'(x) & x \in B_0, \\ E_1\tilde{u}'(x) & x \in B_1, \end{cases} \tag{4.16}$$

the above argument shows that $\tilde{\sigma}(x)$ can vanish *only* if $x \in B_1$. Of course, $\tilde{\sigma}(x)$ is a continuous function, and because of (4.14) it is piecewise affine and decreasing in $B = B_0 \cup B_1$. Note that the function

$$\tilde{G}(x) \equiv \frac{E_1 - E_0}{2E_0E_1}(\tilde{\sigma}(x))^2 - g(\rho_1 - \rho_0)\tilde{u}(x) \tag{4.17}$$

also is continuous in B and that from (4.14) and (4.16) we find

$$\tilde{G}'(x) = -\frac{E_1\rho_1 - E_0\rho_0}{E_0E_1} g\tilde{\sigma}(x).$$

Thus, we see that $\tilde{G}(\cdot)$ is piecewise parabolic, and with the aid of (4.11) we observe that $\tilde{G}(\cdot)$ *is decreasing* (*increasing*) *whenever* $\tilde{\sigma}(x)$ *is positive* (*negative*). Finally, from (4.13), note that

$$\tilde{G}(x)|_{x \in B_0} \geqslant \tilde{G}(x)|_{x \in B_1}. \tag{4.18}$$

In particular, in the case that the given data Q and Δ in $\mathscr{A}$ are such that B_0 is separated from B_1 at two points, the functions $\tilde{\sigma}(x)$ and $\tilde{G}(\cdot)$ will be of a form as illustrated in Figs 4 and 5, respectively. In these figures, the point x_0 at which $\tilde{\sigma}(x_0) = 0$ is located midway between the two points x^* and x^{**} that separate B_0 and B_1. Fundamentally, this fact follows from the inequality (4.18) and the piecewise parabolic nature of $\tilde{G}(\cdot)$. Of course, x^* and x^{**} must be such that $|x^{**} - x^*| = Q$. Finally, the value of $\tilde{\sigma}(x)$ at $x = 0$, which corresponds to the constant K in (4.10), must be determined so that

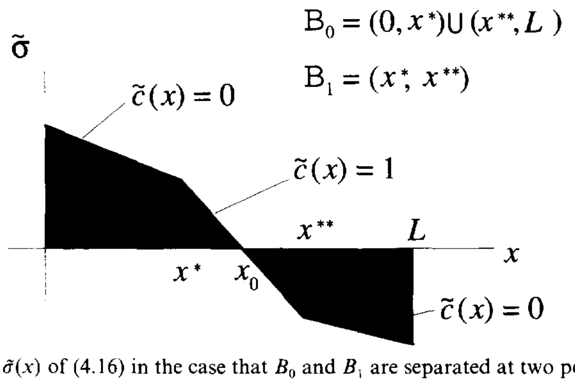

Fig. 4. The stress $\tilde{\sigma}(x)$ of (4.16) in the case that B_0 and B_1 are separated at two points x^* and x^{**}.

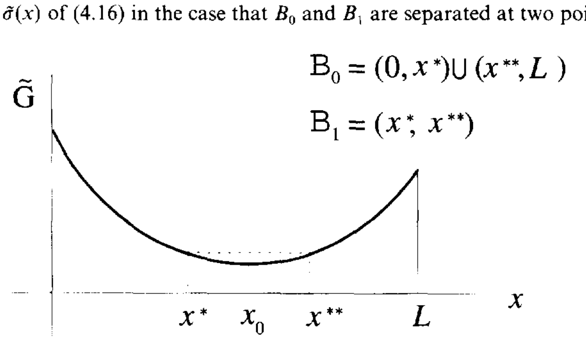

Fig. 5. The function $\tilde{G}(x)$ of (4.17) in the case that B_0 and B_1 are separated at two points x^* and x^{**}.

$$\int_0^L \frac{\tilde{\sigma}(x)}{E^*(\tilde{c}(x))}\,\mathrm{d}x = \int_0^L \tilde{u}'(x)\,\mathrm{d}x = \Delta. \tag{4.19}$$

Then, aside from the determination of an additional constant that fixes $\tilde{u}(0) = 0$, and modulo the adjustments on sets of Lebesgue measure zero as mentioned earlier, the minimizer $(\tilde{c}(\cdot), \tilde{u}(\cdot)) \in \mathscr{A}$ is uniquely determined. In the following section, we summarize the construction of the minimizer $(\tilde{c}(\cdot), \tilde{u}(\cdot)) \in \mathscr{A}$ as a function of Δ.

5. SUMMARY OF THE MINIMIZING FIELD

In describing the character of the minimizer $(\tilde{c}(\cdot), \tilde{u}(\cdot)) \in \mathscr{A}$ of the relaxed problem (4.9) (as well as the original problem (2.6), (2.7), (2.8)) as a function of the prescribed end displacement Δ, it is instructive to concentrate on the behavior of the stress $\tilde{\sigma}(x)$ of (4.16). Starting with large $\Delta > 0$, the arguments that we used to produce Fig. 4 show that as Δ is decreased, the form of $\tilde{\sigma}(x)$ follows the sequence of seven graphs presented in Fig. 6. In these graphs, the value of the minimizing concentration field $\tilde{c}(\cdot)$ in B also is illustrated. Note that first, in Fig. 6(a), there is only one point $x^* \in (0, L)$ which separates B_0 and B_1, and B_1 is adjacent to the end $x = L$. Then as Δ decreases there appears another such point x^{**}, and the region B_1, of high concentration (where $\tilde{c}(x) = 1$) moves into the interior of the bar. As Δ is further decreased, the region B_1, moves toward the end $x = 0$. Then, after a sufficiently large prescribed compression ($\Delta < 0$), the region B_1 of high concentration remains next to the end $x = 0$.

The number Δ_1 that appears in the captions in Fig. 6 corresponds to the particular prescribed end displacement Δ for which $\tilde{\sigma}(L) = 0$. To calculate this, we recall from (4.14) and (4.16) that

$$\tilde{\sigma}(x) = \begin{cases} -g\rho_0 x + K & x \in B_0, \\ -g\rho_1 x + C & x \in B_1, \end{cases} \tag{5.1}$$

where K (see $(4.10)_1$) and C are constants. The condition that $\tilde{\sigma}(L) = 0$ shows that $C = g\rho_1 L$, and the condition that

$$\int_0^L \tilde{c}(x)\,\mathrm{d}x = \int_{x^*}^L \mathrm{d}x = Q$$

yields $x^* = Q - L$. Then, the continuity of $\tilde{\sigma}(x)$ at $x = x^*$ determines the constant K to be

$$K = g[\rho_1 Q + \rho_0(L - Q)],$$

which corresponds to the total weight of the bar B divided by its cross sectional area A, and this is the stress in B at $x = 0$, i.e. $\tilde{\sigma}(0) = K$. Finally, the number Δ_1 is determined from (4.19) to be

$$\Delta_1 = \frac{g(L-Q)}{2E_0}[\rho_1 Q + \rho_0(L-Q)] + \frac{g\rho_1 Q}{2}\left[\frac{Q}{E_1} + \frac{L-Q}{E_0}\right]. \tag{5.2}$$

The number Δ_2 that appears first in Fig. 6(b) corresponds to value of Δ such that if $-\Delta_2 < \Delta < \Delta_2$ then there are two points x^* and x^{**} in B which separate B_0 and B_1. The region of high concentration, where $\tilde{c}(x) = 1$, has length Q and is in an interior part of the bar. This is illustrated in Figs 6(c), (d) and (e). Because in this case we found at the end of Section 4 that the point $x_0 \in B_1$ for which $\tilde{\sigma}(x_0) = 0$ must be located midway between x^* and x^{**}, then at $\Delta = \Delta_2$, when $x^{**} = L$ and $x^* = L - Q$, we must have $x_0 = L - Q/2$. This implies that $C = g\rho_1(L - Q/2)$ in $(5.1)_2$. Then, the continuity of $\tilde{\sigma}(x)$ at $x = x^*$ yields

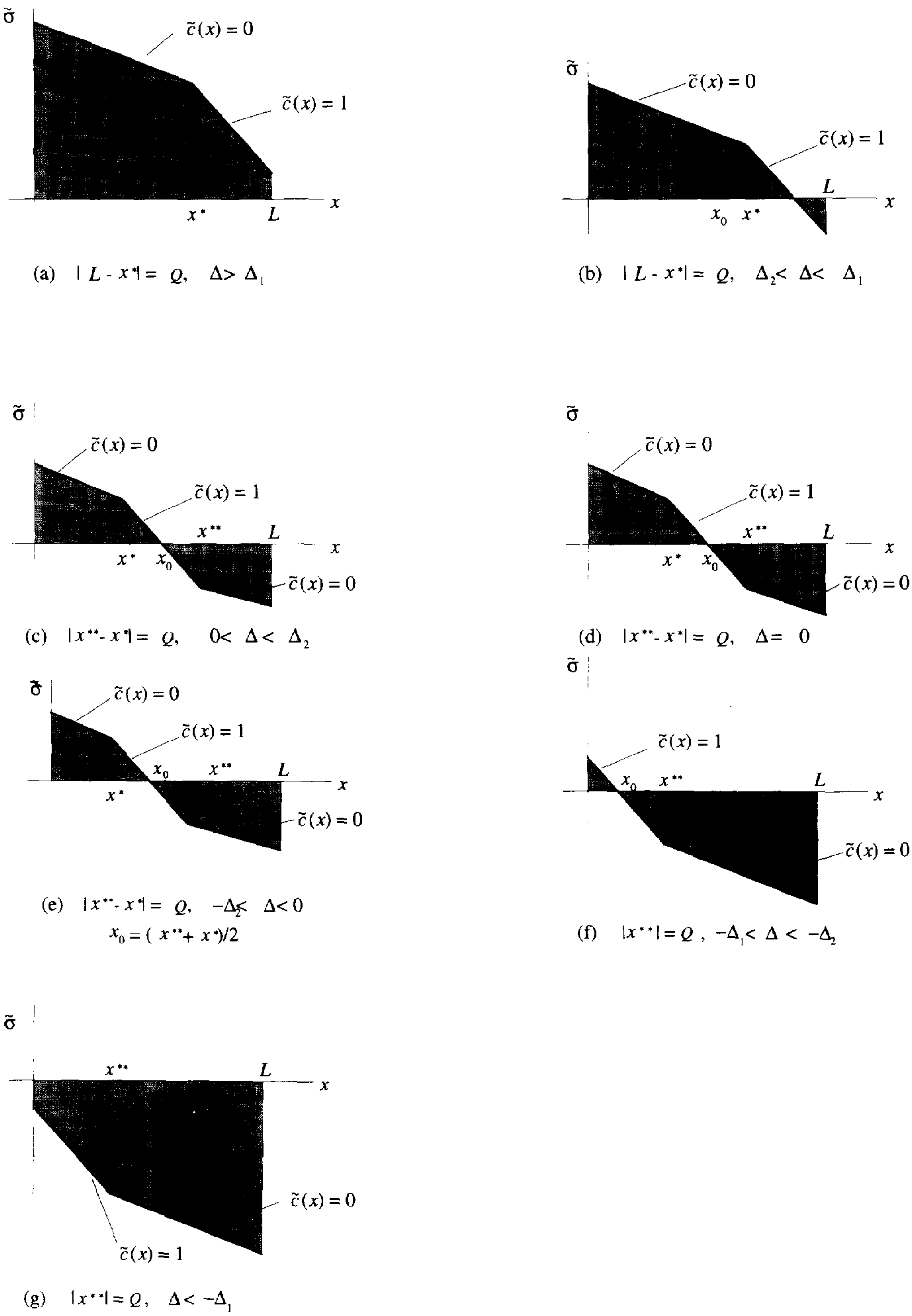

Fig. 6. Summary of the minimizing stress $\tilde{\sigma}(x)$ (see (4.16)) and concentration $\tilde{c}(x)$ in B as a function of the prescribed end displacement Δ.

$$K = g[\rho_1 Q/2 + \rho_0(L-Q)]$$

in $(5.1)_1$. With the use of (4.19), we readily find

$$\Delta_2 = \frac{g(L-Q)}{2E_0}[\rho_1 Q + \rho_0(L-Q)] \tag{5.3}$$

and it is clear from (5.2) that $\Delta_2 < \Delta_1$.

Elementary calculations similar to those indicated above for Δ_1 and Δ_2 show that as long as $\Delta \in (-\Delta_2, \Delta_2)$, there are two points x^* and x^{**} in B which separate B_0 and B_1. Also, we find that for $\Delta < -\Delta_1$ $(< -\Delta_2)$ the stress is everywhere compressive in the bar, as is shown in Fig. 6(g).

In conclusion, we considered a bar which is under the influence of an axial gravitational field and which is fixed at one end and displaced at the other by a given amount. The bar is composed of a binary mixture of two solids. We found that in the state of lowest total potential energy the mixture is uniquely separated and the bar is uniquely distorted. The stiffer material (i.e., the material having greater modulus of elasticity in its pure form) tends to migrate to that interval of the bar that supports the smallest absolute values of stress. In the limit of vanishingly small gravitational field (i.e., $g \to 0$), the resulting limiting state of the bar is unique and corresponds to one of the many arrangements of minimizers that were found in Section 3. Thus, the presence of a gravitational field, no matter how small, provides a sufficient bias to the minimization problem of Section 3 so that a unique minimizer can be identified. For example, if $g > 0$ and $g \to 0$ we find that $(\tilde{c}(x), \tilde{u}'(x))$ limits to the form given in (3.4), (3.5) and (3.2), where B_0 corresponds to the interval $(0, L-Q)$ if $\Delta > 0$ and to the interval (Q, L) if $\Delta < 0$.

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REFERENCES

Dunn, J. E. and Fosdick, R. L. (1980). The morphology and stability of material phases. *Arch. Rational Mech. Anal.* **74,** 1–99.

Fosdick, R. L., Royer, G. F. and Weinberger, H. (1995). A note on the optimal state of a binary solid mixture in a stressed elastic bar. To appear, *Meccanica.*

Buttazzo, G. (1989). *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations.* Pitman Research Notes in Mathematics Series 207. Longman Scientific & Technical Series. John Wiley, New York.

APPENDIX A

One of the fundamental theorems in the calculus of variation (see, e.g., Buttazzo (1989), p. 10) states that if (χ, τ) is a topological space and $F: \chi \to \mathbb{R}$ is a function which is both τ-lower semicontinuous and τ-coercive, then F admits a minimum point on χ. Our aim is here to verify that this theorem implies that the relaxed minimization problem, as defined in (4.7) and (4.8), has a minimizer in $(\varphi_0(\cdot), u_0'(\cdot)) + \bar{\mathscr{A}}_0$ with respect to the topology τ which was introduced following (4.8). To see that the functional $\bar{E}^*(\varphi, u)$ of (4.7) is τ-lower semicontinuous, it suffices to observe that the integrand in (4.7) is convex in the variables $(\varphi'(x), u'(x))$.

We now consider the question of τ-coercivity. Clearly, here it is sufficient to show that for all $(\varphi(\cdot), u(\cdot)) \in (\varphi_0(\cdot), u_0(\cdot)) + \bar{\mathscr{A}}_0$ the functional (4.7) satisfies

$$\bar{E}^*(\varphi, u) \geqslant \alpha \|(\varphi, u)\| + \beta \tag{A1}$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. It is convenient to introduce the substitution variables $\xi_1 = \varphi'$ and $\xi_2 = u'$, and use (4.1) to write the integrand of (4.7) in the form

$$f(x, \varphi, \xi) = \tfrac{1}{2}E^*(\xi_1)\xi_2^2 + g\xi_2(x\rho_0 + (\rho_1 - \rho_0)\varphi),$$

where $\xi = (\xi_1, \xi_2)$. Then, because of the properties of the set $(\varphi_0(\cdot), u_0(\cdot)) + \bar{\mathscr{A}}_0$, we know that $\xi_1 \in [0, 1]$ and $\varphi \in [0, Q]$, and using (4.2) we find that

$$f(x, \varphi, \xi) \geqslant \tfrac{1}{2}E_0\xi_2^2 - \frac{gM}{A}|\xi_2|,$$

where $M = (L\rho_0 + (\rho_1 - \rho_0)Q)A > 0$ is the total mass of the bar B and A is the cross sectional area. Since $\xi_1 \in [0,1]$, it thus follows that

$$f(x, \varphi, \xi) \geqslant a|\xi|^2 + b,$$

for some $a > 0$ and $b \in \mathbb{R}$. Whence, for every $(\varphi(\cdot), u(\cdot)) \in (\varphi_0(\cdot), u_0(\cdot)) + \bar{\mathscr{A}}_0$,

$$\bar{E}^*(\varphi, u) \geqslant a \int_0^L [(\varphi'(x))^2 + (u'(x))^2]\,\mathrm{d}x + bL. \tag{A2}$$

Now, the Poincaré inequality implies that

$$\|u - u_0\|_{L^2(B)} \leqslant C_0(\|u'\|_{L^2(B)} + 1), \quad C_0 > 0,$$

and so we find that

$$\|u\|_{W^{1,2}(B)} \leqslant C_1(\|u'\|_{L^2(B)} + 1)$$

for some $C_1 > 0$. This, together with the definition $\|(\varphi, u)\| = \|\varphi\|_{W^{1,\infty}(B)} + \|u\|_{W^{1,2}(B)}$ and the fact that $\varphi(\cdot) \in W^{1,\infty}(B)$ shows that (A2) implies (A1). Thus the functional $\bar{E}^*(\varphi, u)$ is τ-coercive. Since we know that $\bar{E}^*(\varphi, u)$ is τ-lower semicontinuous then, according to the theorem stated above, the infimum of $\bar{E}^*(\varphi, u)$ over all $(\varphi(\cdot), u(\cdot)) \in (\varphi_0(\cdot), u_0(\cdot)) + \bar{\mathscr{A}}_0$ attains its minimum with respect to the topology τ. Recall that this topology is weak* $W^{1,\infty}(B)$ with respect to the functions $\varphi(\cdot)$ and weak $W^{1,2}(B)$ with respect to the functions $u(\cdot)$.